

# PALEY–WIENER THEOREMS FOR THE $U(n)$ –SPHERICAL TRANSFORM ON THE HEISENBERG GROUP

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ABSTRACT. We prove several Paley–Wiener-type theorems related to the spherical transform on the Gelfand pair  $(H_n \rtimes U(n), U(n))$ , where  $H_n$  is the  $2n+1$ -dimensional Heisenberg group.

Adopting the standard realization of the Gelfand spectrum as the Heisenberg fan in  $\mathbb{R}^2$ , we prove that spherical transforms of  $U(n)$ –invariant functions and distributions with compact support in  $H_n$  admit unique entire extensions to  $\mathbb{C}^2$ , and we find real-variable characterizations of such transforms. Next, we characterize the inverse spherical transforms of compactly supported functions and distributions on the fan, giving analogous characterizations.

## 1. INTRODUCTION

The spherical transform for the Gelfand pair  $(H_n \rtimes U(n), U(n))$  maps  $U(n)$ –invariant functions, i.e. radial functions, on the Heisenberg group  $H_n$  to functions on the Heisenberg fan  $\Sigma$ , which is naturally realized as a closed subset of  $\mathbb{R}^2$ , the *Heisenberg fan* defined in (3.4). In [4, 5] we have studied the image of the space  $\mathcal{S}_{\text{rad}}(H_n)$  of radial Schwartz functions, showing that it consists of the restrictions to  $\Sigma$  of Schwartz functions on  $\mathbb{R}^2$ .

In this paper we first use this result to extend the notion of spherical transform to tempered radial distributions, identifying such transforms as the distributions on  $\mathbb{R}^2$  which are

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“synthetizable” on  $\Sigma$ , i.e., vanish on functions which are identically zero on the fan. Then we prove Paley–Wiener type theorems for the spherical transform  $\mathcal{G}$  and its inverse.

The natural starting point for establishing Paley–Wiener theorems for  $\mathcal{G}$  is the fact that, when  $f$  has compact support, its spherical transform  $\mathcal{G}f$  can be extended from the set of bounded spherical functions (the Gelfand spectrum) to the set of all spherical functions. Spherical functions are parametrized by the pairs  $(\xi, \lambda) \in \mathbb{C}^2$  of their eigenvalues with respect to the two fundamental differential operators,  $L$  (the sublaplacian) and  $i^{-1}T$  (the central derivative). Moreover, the spherical function  $\Phi_{\xi, \lambda}$  with eigenvalues  $(\xi, \lambda) \in \mathbb{C}^2$  depends holomorphically on  $(\xi, \lambda)$ . This allows to extend the spherical transform of a function or distribution with compact support to an entire function on  $\mathbb{C}^2$ .

Symmetrically, each spherical function  $\Phi_{\xi, \lambda}$  extends to an entire function on the complexification  $H_n^{\mathbb{C}}$  of  $H_n$ , and the inversion formula shows that if  $\mathcal{G}f$  has compact support in the Gelfand spectrum, then the function itself extends to an entire function on  $H_n^{\mathbb{C}}$ .

It does not look plausible to have a simple “complex variable” description of the entire functions which are in the range of the spherical, or inverse spherical, transform of the space of  $C^\infty$ -functions, or of distributions, with compact support, see also the comments in Fuhr [16], in a context that is closely related to ours.

We rather look for analogues of the “real variable” characterization of the classical Paley–Wiener spaces in  $\mathbb{R}^n$ , in the spirit of the works of Bang [7] and Tuan [25], later expanded and refined by Andersen and deJeu [1]. We take the following as our model statement [1]: a function  $f$  on  $\mathbb{R}^n$  is the Fourier transform of a  $C^\infty$  function with compact support if and only if it is a Schwartz function and, for some  $p \in [1, \infty]$ ,

$$(1.1) \quad \limsup_{k \rightarrow \infty} \|\Delta^k f\|_p^{\frac{1}{k}} < \infty .$$

In this case, the left-hand side is finite for every  $p$ , the “lim sup” is a limit and, for every  $p \in [1, \infty]$ ,

$$\lim_{k \rightarrow \infty} \|\Delta^k f\|_p^{\frac{1}{k}} = \max_{x \in \text{supp } \mathcal{F}^{-1} f} |x|^2 .$$

When restricted to radial functions, this theorem can be reformulated in terms of the spherical transform  $\mathcal{G}$  for the Gelfand pair  $(\mathbb{R}^n \rtimes \text{SO}_n, \text{SO}_n)$ , given by  $\mathcal{G}f(\lambda) = \hat{f}(\xi)$  with  $|\xi|^2 = \lambda \geq 0$ . Then we have two different statements, depending on the side of the Fourier transform it is applied on:

- (i) a function  $g$  on the Gelfand spectrum  $[0, +\infty)$  is the spherical transform of a radial  $C^\infty$  function on  $\mathbb{R}^n$  with compact support if and only if it is a Schwartz function and, for some  $p \in [1, \infty]$ ,

$$\limsup_{k \rightarrow \infty} \|g^{(k)}\|_p^{\frac{1}{k}} < \infty ;$$

in this case, for every  $p \in [1, \infty]$ ,

$$\lim_{k \rightarrow \infty} \|g^{(k)}\|_p^{\frac{1}{k}} = \max_{x \in \text{supp } \mathcal{G}^{-1} g} |x|^2 .$$

- (ii) a radial function  $f$  on  $\mathbb{R}^n$  is the inverse spherical transform of a  $C^\infty$  function with compact support in  $[0, +\infty)$  if and only if it is a Schwartz function and (1.1) holds for some  $p \in [1, \infty]$ ; in this case, for every  $p \in [1, \infty]$ ,

$$\lim_{k \rightarrow \infty} \|\Delta^k f\|_p^{\frac{1}{k}} = \max_{\lambda \in \text{supp } \mathcal{G} f} \lambda .$$

We regard (i) as a Paley-Wiener theorem for the (direct) spherical transform, and (ii) as a Paley-Wiener theorem for the inverse spherical transform.

Possible analogues of (i) and (ii) for the pair  $(H_n \rtimes \text{U}(n), \text{U}(n))$  rely on the identification, for each direction, of a differential operator on one side of the spherical transforms and the corresponding “norm” on the other side.

Our results are related to the following choices:

- (i') the difference/differential operators  $M_{\pm}$  of Benson, Jenkins and Ratcliff [8] on  $\Sigma$  and the Korányi norm (2.2) on  $H_n$ ;
- (ii') the sublaplacian on  $H_n$  and its eigenvalue  $\xi$  on  $\Sigma$ .

We first prove real Paley-Wiener theorems for the direct spherical transform, i.e. analogues of (i) with the ingredients in (i'). We treat the cases of  $C^{\infty}$  and  $L^2$  functions and of tempered distributions. These characterizations are summarized in Theorem 7.1, Corollary 7.7 and Theorem 7.8.

We also remark that the (unique) entire extension of the transform of a function in  $\mathcal{D}_{\text{rad}}(H_n)$  needs not be Schwartz on  $\mathbb{R}^2$ . This shows that, in general, the Schwartz extensions to  $\mathbb{R}^2$  constructed in [5] are different from the entire extension discussed here.

In the second part of the paper, we show that, given a distribution  $U$  on  $\mathbb{R}^2$  with compact support, the inversion formula for the spherical transform produces a function on the Heisenberg group  $H_n \simeq \mathbb{R}^{2n+1}$  which can be analytically extended to  $\mathbb{C}^{2n+1}$ . If the distribution  $U$  is synthetizable on  $\Sigma$ , the function so obtained on  $H_n$  coincides with its inverse spherical transform. For such distributions  $U$ , we obtain a real Paley-Wiener analogue of (ii) with the ingredients in (ii'). A similar theorem is also proved for functions on  $\Sigma$  which are either restrictions of  $C^{\infty}$  functions or are square integrable with respect to the Plancherel measure. Our characterizations are summarized in Theorem 8.4, Theorem 8.7 and Theorem 8.9. These results can be interpreted as a “real” spectral Paley-Wiener theorems for the spectral measure of the sublaplacian, a point of view which coincides with that of [16].

There is a wide literature on Paley–Wiener theorems on the Heisenberg group. The earliest result is due to Ando [2], followed by Thangavelu [21, 22, 23], Arnal and Ludwig [3], Narayanan and Thangavelu [19]. Results are mostly related to the group (operator-valued) Fourier transform and its inverse, but there are also “spectral” Paley–Wiener theorems, as in

the already mentioned paper [16], where the condition of compact support on the transform of a given function is replaced by the condition that the function itself belongs to the image of the spectral measure of a compact set in  $\mathbb{R}^+$  associated to the sublaplacian (see also Strichartz [20], Bray [10], and Dann and Ólafsson [12] in other contexts).

Our paper is organized as follows. In Section 2 we introduce the basic notation. In Section 3 we treat spherical functions noting that they can be extended to holomorphic functions in each variable and providing some easy estimates. Section 4 and Section 5 deal with the spherical transform of radial functions and radial tempered distributions, respectively. In Section 6 we prove some properties of the operators  $M_{\pm}$  first introduced in [8]. These are exploited in Sections 7 and 8 to obtain real Paley–Wiener theorems for the spherical transform and its inverse, respectively.

## 2. NOTATION

We denote by  $H_n$  the Heisenberg group, i.e., the real manifold  $\mathbb{C}^n \times \mathbb{R}$  equipped with the group law

$$(z, t)(w, u) = \left(z + w, t + u + \frac{1}{2} \operatorname{Im} \langle w | z \rangle\right) \quad \forall z, w \in \mathbb{C}^n, \quad t, u \in \mathbb{R},$$

where  $\langle \cdot | \cdot \rangle$  denotes the Hermitian inner product in  $\mathbb{C}^n$ .

It is easy to check that the Lebesgue measure  $dm = dz dt$  is a Haar measure on  $H_n$ .

We denote by  $T$ ,  $Z_j$  and  $\bar{Z}_j$ , where  $j = 1, \dots, n$ , the left-invariant vector fields

$$Z_j = \partial_{z_j} - \frac{i}{4} \bar{z}_j \partial_t \quad \bar{Z}_j = \partial_{\bar{z}_j} + \frac{i}{4} z_j \partial_t, \quad T = \partial_t.$$

The only nontrivial brackets are  $T = -2i [Z_j, \bar{Z}_j]$ .

The operators  $Z_j$  and  $\bar{Z}_j$  are homogeneous of degree 1 while  $T$  is homogeneous of degree 2 with respect to the anisotropic dilations  $r \cdot (z, t) = (rz, r^2t)$ , where  $r > 0$  and  $(z, t) \in H_n$ .

Let  $I = (i_1, \dots, i_{2n+1})$  be in  $\mathbb{N}^{2n+1}$ ; we denote by  $D^I$  a differential operator of homogeneous degree  $\deg I = i_1 + \dots + i_{2n} + 2i_{2n+1}$  of the form

$$(2.1) \quad D^I = Z_1^{i_1} \bar{Z}_1^{i_2} \dots Z_n^{i_{2n-1}} \bar{Z}_n^{i_{2n}} T^{i_{2n+1}}.$$

The monomials  $D^I$  with  $\deg I = j$  form a basis of the space of all left-invariant differential operators on  $H_n$  which are homogeneous of degree  $j$ .

We write  $\mathcal{S}(H_n)$  for the Schwartz space of functions on  $H_n$ , i.e., the space of infinitely differentiable functions  $f$  on  $H_n$  such that all partial derivatives  $D^I f$  of  $f$  are rapidly decreasing. The Schwartz space is equipped with the following family of norms, parametrized by a nonnegative integer  $p$ :

$$\|f\|_{(p)} = \sup_{(z,t) \in H_n} \{(1 + |(z,t)|)^p |D^I f(z,t)| : \deg I \leq p\},$$

where

$$(2.2) \quad |(z,t)| = \left( \frac{|z|^4}{16} + t^2 \right)^{1/4}.$$

We also define  $\mathcal{A}$  as

$$\mathcal{A}(z,t) = \frac{|z|^2}{4} + it \quad \forall (z,t) \in H_n,$$

so that  $|A(z,t)| = |(z,t)|^2$ .

### 3. SPHERICAL FUNCTIONS

The unitary group  $U(n)$  acts on  $H_n$  via

$$k \cdot (z,t) = (kz,t) \quad \forall (z,t) \in H_n, \quad k \in U(n).$$

This action induces an action on functions  $f$  on  $H_n$  by the formula

$$k \cdot f(z,t) = f(k^{-1}z,t) \quad \forall k \in U(n), \quad (z,t) \in H_n.$$

We note that a function  $f$  on  $H_n$  is  $U(n)$ –invariant if and only if it depends only on  $|z|$  and  $t$ , therefore we shall call it radial. We denote by  $\mathcal{S}_{\text{rad}}(H_n)$  the space of radial Schwartz functions.

Denote by  $G$  the semidirect product  $H_n \rtimes U(n)$ . We may identify the space of smooth bi- $U(n)$ –invariant functions  $\mathcal{D}(G//U(n))$  with the algebra  $\mathcal{D}_{\text{rad}}(H_n)$  of smooth radial functions on  $H_n$  with compact support. It is known [17, 11] that  $(G, U(n))$  is a Gelfand pair, i.e.,  $\mathcal{D}_{\text{rad}}(H_n)$  is a commutative algebra. We may also identify the commutative algebra  $\mathbb{D}(G/U(n))$  of  $G$ –invariant differential operators on  $G/U(n)$  with the algebra  $\mathbb{D}_{\text{rad}}$  of all left-invariant and  $U(n)$ –invariant differential operators on  $H_n$ , which has two essentially self-adjoint generators, namely  $i^{-1}T$  and the sublaplacian

$$L = -2 \sum_{j=1}^n (Z_j \bar{Z}_j + \bar{Z}_j Z_j).$$

The spherical functions are characterized as the joint eigenfunctions of all  $G$ –invariant differential operators on  $G/U(n)$ , i.e., as the radial eigenfunctions of  $i^{-1}T$  and  $L$ , normalized to take value 1 at identity. Spherical functions are analytic and are uniquely determined by the pair of their eigenvalues relative to  $L$  and  $i^{-1}T$  respectively.

The next subsection shows that the spherical function  $\Phi_{\xi, \lambda}$  exists for every pair  $(\xi, \lambda)$  of eigenvalues and that depends holomorphically on variables and parameters.

**3.1. Holomorphy of spherical functions.** We initially consider real eigenvalues  $\xi$  and  $\lambda$  and look for a radial solution of the system

$$(3.1) \quad \begin{cases} Lu = \xi u \\ Tu = i\lambda u \\ u(0, 0) = 1 . \end{cases}$$

Following [18], for  $\lambda \neq 0$  we write the solution in the form

$$u(z, t) = e^{i\lambda t} e^{-\lambda|z|^2/4} v(\lambda|z|^2/2),$$

obtaining that  $v$  satisfies the confluent hypergeometric differential equation

$$(3.2) \quad s v''(s) + (c - s)v'(s) - a v(s) = 0$$

with parameters  $a = \frac{n}{2} - \frac{\xi}{2\lambda}$ ,  $c = n$ . The normalized solution of (3.2) is the confluent hypergeometric function

$${}_1F_1(a, c; s) = 1 + \frac{a}{c} s + \frac{a(a+1)}{c(c+1)} \frac{s^2}{2!} + \cdots = \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} \frac{s^k}{k!},$$

where  $(a)_0 = 1$ ,  $(a)_k = \Gamma(a+k)/\Gamma(a)$ , so that for real  $\lambda \neq 0$

$$u(z, t) = e^{i\lambda t} e^{-\lambda|z|^2/4} {}_1F_1\left(\frac{n}{2} - \frac{\xi}{2\lambda}, n; \lambda|z|^2/2\right).$$

When  $\lambda = 0$  and  $\xi$  is real, a similar procedure shows that

$$u(z, t) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(n)_k} \frac{(\xi|z|^2/4)^k}{k!} = \mathcal{J}_{n-1}(\xi|z|^2/4) \quad \forall (z, t) \in H_n,$$

where

$$\mathcal{J}_{\beta}(s) = \sum_{k=0}^{+\infty} \frac{(-s)^k}{k! (\beta+1)_k} \quad \forall s \in \mathbb{C}.$$

Note that  $J_{\beta}(u) = \frac{(u/2)^{\beta}}{\beta!} \mathcal{J}_{\beta}(u^2/4)$  is the Bessel function of the first kind of order  $\beta$ .

Therefore for every pair of real numbers  $\xi$  and  $\lambda$  we have the spherical function

$$\Phi_{\xi, \lambda}(z, t) = \begin{cases} e^{i\lambda t} e^{-\lambda|z|^2/4} {}_1F_1\left(\frac{n}{2} - \frac{\xi}{2\lambda}, n; \frac{\lambda|z|^2}{2}\right) & \lambda \neq 0 \\ \mathcal{J}_{n-1}(\xi|z|^2/4) & \lambda = 0 \end{cases} \quad \forall (z, t) \in H_n.$$

We now verify that  $\lambda \mapsto \Phi_{\xi, \lambda}(z, t)$  is regular in  $\lambda = 0$ . Indeed, something more holds.



**Lemma 3.1.** *The function  $(x, y, t, \xi, \lambda) \mapsto \Phi_{\xi, \lambda}(x + iy, t)$  extends to a holomorphic function on  $\mathbb{C}^{2n+3}$ .*

*Proof.* Note that when  $\lambda \neq 0$ ,  $z = x + iy$ ,

$$\begin{aligned} {}_1F_1\left(\frac{n\lambda - \xi}{2\lambda}, n; \frac{\lambda|z|^2}{2}\right) &= \sum_{k=0}^{\infty} \frac{\left(\frac{n\lambda - \xi}{2\lambda}\right)_k}{(n)_k k!} \left(\frac{\lambda|z|^2}{2}\right)^k \\ &= 1 + \sum_{k=1}^{\infty} \frac{(x^2 + y^2)^k}{(n)_k k! 4^k} \prod_{d=0}^{k-1} (\lambda(2d + n) - \xi), \end{aligned}$$

so that for all  $(\xi, \lambda, x, y, t)$  the function

$$(3.3) \quad \Phi_{\xi, \lambda}(x + iy, t) = e^{i\lambda t} e^{-\lambda(x^2 + y^2)/4} \left(1 + \sum_{k=1}^{\infty} \frac{(x^2 + y^2)^k}{(n)_k k! 4^k} \prod_{d=0}^{k-1} (\lambda(2d + n) - \xi)\right)$$

is a series of entire functions converging uniformly on compact sets.  $\square$

By analytic continuation,  $\Phi_{\xi, \lambda}$  is the spherical function for every  $(\xi, \lambda) \in \mathbb{C}^2$ .

**3.2. Bounded spherical functions.** The Gelfand spectrum of the Banach algebra  $L_{\text{rad}}^1(H_n)$  of radial integrable functions is given by the set of normalized bounded spherical functions, equipped with the compact open topology. We recall [18] that  $\Phi_{\xi, \lambda}$  is bounded on  $H_n$  if and only if  $(\xi, \lambda)$  belongs to the so called Heisenberg fan given by

$$(3.4) \quad \Sigma = \Sigma^* \cup \{(\xi, 0) \in \mathbb{R}^2 : \xi \geq 0\},$$

where

$$\Sigma^* = \{(\xi, \lambda) \in \mathbb{R}^2 : \lambda \neq 0, \xi = |\lambda|(2j + n), j \in \mathbb{N}\}.$$

It is known that  $\Sigma$  is homeomorphic to the Gelfand spectrum [14, 9].

When  $(\xi, \lambda)$  is in  $\Sigma^*$ , the spherical function  $\Phi_{\xi, \lambda}$  can be written in terms of Laguerre polynomials, which is the form that we usually find in literature. Indeed, the relation (see

[13, p. 253, formula (7)]

$$(3.5) \quad {}_1F_1(a, n; x) = e^x {}_1F_1(n - a, n; -x),$$

implies that

$$\Phi_{\xi, \lambda}(z, 0) = \Phi_{\xi, |\lambda|}(z, 0) = e^{-|\lambda||z|^2/4} {}_1F_1\left(\frac{n}{2} - \frac{\xi}{2|\lambda|}, n; \frac{|\lambda||z|^2}{2}\right)$$

and when  $\xi = |\lambda|(2j + n)$  the hypergeometric function in the previous formula coincides with the normalized  $j^{\text{th}}$  Laguerre polynomial of order  $n - 1$ , i.e.,

$${}_1F_1\left(-j, n; \frac{|\lambda||z|^2}{2}\right) = \frac{1}{\binom{j+n-1}{j}} \sum_{k=0}^j \binom{j+\beta}{j-k} \frac{\left(-\frac{|\lambda||z|^2}{2}\right)^k}{k!}.$$

**3.3. Estimates of derivatives of spherical functions.** In this subsection we exploit the fact that the bounded spherical functions  $\{\Phi_{\xi, \lambda}\}_{(\xi, \lambda) \in \Sigma}$  are averages of coefficients of irreducible unitary representations of  $H_n$  to give some estimates that we shall need in the sequel. Referring to the Bargmann-Fock model of the irreducible representations  $\pi^\lambda$  of  $H_n$  associated to the character  $e^{i\lambda t}$  on the center, we represent the operators  $\pi^\lambda(z, t)$  as matrices  $(\pi_{\mathbf{j}, \mathbf{k}}^\lambda(z, t))_{\mathbf{j}, \mathbf{k} \in \mathbb{N}^n}$  in the basis of normalized monomials (for more details see the monographs [15] or [24]). Then the bounded spherical functions can be written as averages of diagonal entries of this matrix according to the rule

$$(3.6) \quad \Phi_{\xi, \lambda} = \frac{1}{\binom{j+n-1}{j}} \sum_{|\mathbf{j}|=j} \pi_{\mathbf{j}, \mathbf{j}}^\lambda,$$

where  $\xi = |\lambda|(2j + n)$  and  $|\mathbf{j}| = \mathbf{j}_1 + \dots + \mathbf{j}_n$  for  $\mathbf{j} = (\mathbf{j}_1, \dots, \mathbf{j}_n) \in \mathbb{N}^n$ .

**Lemma 3.2.** *Let  $D^I$  be a differential operator of homogeneous degree  $\deg I = \alpha$  as in (2.1).*

*Then*

$$|D^I \Phi_{\xi, \lambda}(z, t)| \leq C_\alpha (1 + \xi)^{\alpha/2} \quad \forall (\xi, \lambda) \in \Sigma, (z, t) \in H_n.$$

*Proof.* Let  $\lambda \neq 0$ . Here and afterwards, if any component of the multiindices  $\mathbf{j}$  or  $\mathbf{k}$  is negative, then  $\pi_{\mathbf{j},\mathbf{k}}^\lambda = 0$ . Since the representations are unitary,  $|\pi_{\mathbf{j},\mathbf{k}}^\lambda| \leq 1$  and it is easy to check that

$$Z_i \pi_{\mathbf{j},\mathbf{k}}^\lambda = \begin{cases} -\sqrt{\frac{k_i \lambda}{2}} \pi_{\mathbf{j},\mathbf{k}-\mathbf{e}_i}^\lambda & \lambda > 0 \\ \sqrt{\frac{(k_i+1)|\lambda|}{2}} \pi_{\mathbf{j},\mathbf{k}+\mathbf{e}_i}^\lambda & \lambda < 0 \end{cases} \quad \bar{Z}_i \pi_{\mathbf{j},\mathbf{k}}^\lambda = \begin{cases} \sqrt{\frac{(k_i+1)\lambda}{2}} \pi_{\mathbf{j},\mathbf{k}+\mathbf{e}_i}^\lambda & \lambda > 0 \\ -\sqrt{\frac{k_i|\lambda|}{2}} \pi_{\mathbf{j},\mathbf{k}-\mathbf{e}_i}^\lambda & \lambda < 0 \end{cases}$$

and  $T \pi_{\mathbf{j},\mathbf{j}}^\lambda = i \lambda \pi_{\mathbf{j},\mathbf{j}}^\lambda$ . Here  $\mathbf{e}_i$  is the multiindex with just the  $i^{th}$  component equal to 1.

Suppose that  $(\xi, \lambda)$  is in  $\Sigma^*$ , with  $\xi = |\lambda|(2|\mathbf{j}| + n)$ . Then if  $D^I = Z^{\mathbf{k}} \bar{Z}^{\mathbf{h}} T^s$  with  $\deg I = \alpha = |\mathbf{k}| + |\mathbf{h}| + 2s$  we have

$$\begin{aligned} |D^I \pi_{\mathbf{j},\mathbf{j}}^\lambda| &= |\lambda|^s |Z^{\mathbf{k}} \bar{Z}^{\mathbf{h}} \pi_{\mathbf{j},\mathbf{j}}^\lambda| \\ &= |\lambda|^s \begin{cases} \sqrt{\prod_{i=1}^n \prod_{\ell=1}^{h_i} \frac{\lambda}{2} (j_i + \ell)} \sqrt{\prod_{i=1}^n \prod_{\ell=1}^{k_i} \frac{\lambda}{2} (j_i + h_i + 1 - \ell)} |\pi_{\mathbf{j},\mathbf{j}+\mathbf{h}-\mathbf{k}}^\lambda| & \lambda > 0 \\ \sqrt{\prod_{i=1}^n \prod_{\ell=1}^{h_i} \frac{|\lambda|}{2} (j_i + 1 - \ell)} \sqrt{\prod_{i=1}^n \prod_{\ell=1}^{k_i} \frac{|\lambda|}{2} (j_i - h_i + \ell)} |\pi_{\mathbf{j},\mathbf{j}-\mathbf{h}+\mathbf{k}}^\lambda| & \lambda < 0 \end{cases} \\ &\leq C |\lambda|^s \sqrt{|\lambda|^{|\mathbf{h}|+|\mathbf{k}|} (|\mathbf{j}| + |\mathbf{h}|)^{|\mathbf{h}|} (|\mathbf{j}| + |\mathbf{h}| + |\mathbf{k}|)^{|\mathbf{k}|}} \\ &\leq C_\alpha (1 + \xi)^{\alpha/2}. \end{aligned}$$

Here we have used the fact that in  $\Sigma^*$  we have  $\xi = |\lambda|(2|\mathbf{j}| + n) \geq |\lambda|$ . By (3.6) the thesis follows on  $\Sigma^*$ , and by continuity the same estimates hold on  $\Sigma$ , thus proving the lemma.  $\square$

#### 4. SPHERICAL TRANSFORM

As usual, we denote by  $\langle \cdot, \cdot \rangle_{H_n}$  the dual pairing on  $H_n$  and we shall also write

$$\langle f, g \rangle_{H_n} = \int_{H_n} f(z, t) g(z, t) dz dt, \quad \forall f, g \in \mathcal{S}(H_n).$$

Given a measurable function  $f$  on  $H_n$  we denote by  $\check{f}$  the function defined by  $\check{f}(x) = f(x^{-1})$  for every  $x$  in  $H_n$ .

**4.1. Definitions and main facts.** Let  $f$  be in  $L^1_{\text{rad}}(H_n)$ . We define its spherical transform  $\mathcal{G}f$  by

$$\mathcal{G}f(\xi, \lambda) = \int_{H_n} f(x) \Phi_{\xi, \lambda}(x^{-1}) dx = \langle f, \check{\Phi}_{\xi, \lambda} \rangle_{H_n} \quad \forall (\xi, \lambda) \in \Sigma.$$

Then  $\mathcal{G}f$  is a continuous function on  $\Sigma$ .

The inversion formula for a function  $f$  in  $\mathcal{S}_{\text{rad}}(H_n)$  is

$$f(x) = \int_{\Sigma} \mathcal{G}f(\xi, \lambda) \Phi_{\xi, \lambda}(x) d\mu(\xi, \lambda) \quad \forall x \in H_n,$$

where  $\mu$  is the Plancherel measure defined by

$$\int_{\Sigma} \psi d\mu = \frac{1}{(2\pi)^{n+1}} \int_{\mathbb{R}} \sum_{j=0}^{\infty} \binom{j+n-1}{j} \psi(|\lambda|(2j+n), \lambda) |\lambda|^n d\lambda \quad \forall \psi \in C_c(\Sigma).$$

It is easy to check that the function  $(\xi, \lambda) \mapsto (1 + \xi)^{-(n+2)}$  is in  $L^1(\Sigma)$ , so

$$(4.1) \quad \|\psi\|_{L^1(\Sigma)} \leq C \|(1 + |\xi|)^{n+2} \psi\|_{L^\infty(\mathbb{R}^2)} \quad \forall \psi \in \mathcal{S}(\mathbb{R}^2).$$

As in [4], we denote by  $\mathcal{S}(\Sigma)$  the space of restrictions to  $\Sigma$  of Schwartz functions on  $\mathbb{R}^2$ , endowed with the quotient topology  $\mathcal{S}(\mathbb{R}^2)/\{\phi : \phi|_{\Sigma} = 0\}$ . For radial Schwartz functions on  $H_n$ , we have proved the following result.

**Theorem 4.1.** [4, Corollary 1.2] *The spherical transform is a topological isomorphism between the spaces  $\mathcal{S}_{\text{rad}}(H_n)$  and  $\mathcal{S}(\Sigma)$ .*

On the other hand, Lemma 3.1 implies that when  $f$  is compactly supported we can regard  $\mathcal{G}f$  as a function on  $\mathbb{C}^2$ .

**Proposition 4.2.** *If  $f$  is in  $\mathcal{D}_{\text{rad}}(H_n)$  then  $\mathcal{G}f$  extends to the holomorphic function  $F$  on  $\mathbb{C}^2$  given by the rule*

$$F(\xi, \lambda) = \langle f, \check{\Phi}_{\xi, \lambda} \rangle_{H_n} \quad \forall (\xi, \lambda) \in \mathbb{C}^2.$$

**4.2. Holomorphic versus Schwartz extensions.** Given  $f$  in  $\mathcal{D}_{\text{rad}}(H_n)$ , we have found two ways of extending its spherical transform  $\mathcal{G}f$  to a smooth function on  $\mathbb{R}^2$ . Namely, by Theorem 4.1, there exists a Schwartz function  $G$  on  $\mathbb{R}^2$  such that  $G|_{\Sigma} = \mathcal{G}f$ , and by Proposition 4.2 the function  $F$  is the holomorphic extension of  $\mathcal{G}f$  to  $\mathbb{C}^2$ . So  $G|_{\Sigma} = F|_{\Sigma} = \mathcal{G}f$ .

We observe that any two entire functions on  $\mathbb{C}^2$ , which coincide on  $\Sigma$ , are everywhere equal, so  $F$  is the unique continuation of  $\mathcal{G}f$  to an entire function on  $\mathbb{C}^2$ .

A question arises naturally: if  $f$  is in  $\mathcal{D}_{\text{rad}}(H_n)$ , is it true that  $F$ , when restricted to real values of  $(\xi, \lambda)$ , is a Schwartz function on  $\mathbb{R}^2$ ?

In the rest of this subsection we show that the answer can be negative.

Let  $f$  be a function of the form

$$f(z, t) = g(z) \otimes h(t) \quad \forall (z, t) \in H_n$$

where  $h$  is even and compactly supported and  $g$  is nonpositive and supported in  $1 < |z| < 4$ , equal to  $-1$  when  $2 < |z| < 3$ .

Let  $\mathcal{F}h$  be the Euclidean Fourier transform of  $h$ . We now show that the function  $\lambda \geq 0 \mapsto |\mathcal{F}h(\lambda)| e^{\lambda/2}$  is not bounded. Indeed, since  $h$  is even, if it were bounded, then for every  $b$ ,  $0 \leq b < 1/2$ , the function  $\lambda \mapsto e^{b|\lambda|} \mathcal{F}h(\lambda)$  would be in  $L^2(\mathbb{R})$ . By the Paley–Wiener Theorem for the Euclidean transform  $h = \mathcal{F}^2 h$  would continue analytically to  $\{w : |\text{Im}(w)| < 1/2\}$ , but this cannot be true since  $h$  has compact support. Therefore the function  $\lambda \geq 0 \mapsto |\mathcal{F}h(\lambda)| e^{\lambda/2}$  is not bounded.

Now, if  $(\xi, \lambda) \in \mathbb{R}^2 \mapsto F(\xi, \lambda)$  were rapidly decreasing, then the same would hold true for the function  $\lambda \mapsto F((n+1)\lambda, \lambda)$ . Note that when  $\lambda > 0$

$$F((n+1)\lambda, \lambda) = \mathcal{F}h(\lambda) \int_{1 < |z| < 4} g(z) e^{-\lambda|z|^2/4} {}_1F_1(-\tfrac{1}{2}, n, \lambda|z|^2/2) dz.$$

Moreover  ${}_1F_1(-\frac{1}{2}, n, x) \leq 0$  when  $x \geq 2n$  and by the estimate (see [13, p. 27, formula (3)])

$${}_1F_1(a, n; x) = \frac{\Gamma(n)}{\Gamma(a)} e^x x^{a-n} (1 + O(x^{-1})), \quad \operatorname{Re}(x) \rightarrow \infty, \quad a \neq 0, -1, -2, \dots,$$

when  $\lambda \rightarrow +\infty$  we obtain

$$\begin{aligned} |F((n+1)\lambda, \lambda)| &= |\mathcal{F}h(\lambda)| \int_{1 < |z| < 4} g(z) e^{-\lambda|z|^2/4} {}_1F_1(-\frac{1}{2}, n, \lambda|z|^2/2) dz \\ &\geq C |\mathcal{F}h(\lambda)| \int_{2 < |z| < 3} e^{-\lambda|z|^2/4} e^{\lambda|z|^2/2} (\lambda|z|^2/2)^{-1/2-n} dz \\ &\geq C |\mathcal{F}h(\lambda)| e^{\lambda/2}, \end{aligned}$$

so the function  $\lambda \mapsto F((n+1)\lambda, \lambda)$  is not bounded.

## 5. THE SPHERICAL TRANSFORM OF RADIAL TEMPERED DISTRIBUTIONS

As usual, we denote by  $\langle \cdot, \cdot \rangle_{\mathbb{R}^2}$  the dual pairing on  $\mathbb{R}^2$  and we also write

$$\langle \varphi, \psi \rangle_{\mathbb{R}^2} = \int_{\mathbb{R}^2} \varphi(x) \psi(x) dx \quad \forall \varphi, \psi \in \mathcal{S}(\mathbb{R}^2).$$

Let  $\Pi : \mathcal{S}(H_n) \longrightarrow \mathcal{S}(H_n)$  be the averaging projector defined by

$$\Pi f = \int_{\mathbf{U}(n)} f \circ k \, dk \quad \forall f \in \mathcal{S}(H_n).$$

Then the Schwartz space on  $H_n$  decomposes into the direct sum  $\mathcal{S}(H_n) = \mathcal{S}_{\text{rad}}(H_n) \oplus \ker \Pi$  so that  $\mathcal{S}_{\text{rad}}(H_n)$  is isomorphic to the quotient space  $\mathcal{S}(H_n)/\ker \Pi$ . It follows that the dual space  $(\mathcal{S}_{\text{rad}}(H_n))'$  is isomorphic to the subspace  $\mathcal{S}'_{\text{rad}}(H_n)$  of  $\mathcal{S}'(H_n)$  consisting of all tempered distributions  $\Lambda$  on  $H_n$  such that

$$\langle \Lambda, f \rangle_{H_n} = 0 \quad \forall f \in \ker \Pi.$$

On the other hand the dual space of  $\mathcal{S}(\Sigma)$  is naturally isomorphic to the subspace  $\mathcal{S}'_0(\Sigma)$  of  $\mathcal{S}'(\mathbb{R}^2)$  consisting of all tempered distributions  $U$  on  $\mathbb{R}^2$  such that

$$\langle U, g \rangle_{\mathbb{R}^2} = 0, \quad \forall g \in \mathcal{S}(\mathbb{R}^2) \text{ such that } g = 0 \text{ on } \Sigma.$$

We note that the Plancherel formula can be written as

$$\langle f, \bar{g} \rangle_{H_n} = \langle \mathcal{G}f \mu, \overline{\mathcal{G}g} \rangle_{\mathbb{R}^2} = \langle \mathcal{G}f \mu, \mathcal{G}\bar{g} \rangle_{\mathbb{R}^2} \quad \forall f, g \in \mathcal{S}_{\text{rad}}(H_n).$$

Therefore we are led to define the spherical transform of a radial tempered distribution  $\Lambda$  on  $H_n$  as the distribution  $\mathcal{G}\Lambda$  in  $\mathcal{S}'(\mathbb{R}^2)$  given by

$$(5.1) \quad \langle \mathcal{G}\Lambda, \varphi \rangle_{\mathbb{R}^2} = \langle \Lambda, (\mathcal{G}^{-1}\varphi|_{\Sigma})^\sim \rangle_{H_n} \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^2).$$

Clearly, if  $\Lambda$  is in  $\mathcal{S}'_{\text{rad}}(H_n)$ , then for every function  $\varphi$  in  $\mathcal{S}(\mathbb{R}^2)$  such that  $\varphi = 0$  on  $\Sigma$  we have  $\langle \mathcal{G}\Lambda, \varphi \rangle_{\mathbb{R}^2} = \langle \Lambda, (\mathcal{G}^{-1}\varphi|_{\Sigma})^\sim \rangle_{H_n} = 0$ , i.e.,  $\mathcal{G}\Lambda$  is in  $\mathcal{S}'_0(\Sigma)$ . We recall that we have denoted by  $m$  the Lebesgue measure on  $H_n$ . If  $f$  is a radial function in  $L^1 \cap L^2(H_n)$ , then  $fm$  is in  $\mathcal{S}'_{\text{rad}}(H_n)$  and  $\mathcal{G}(fm) = (\mathcal{G}f)\mu$  where  $\mathcal{G}f$  has been defined in Subsection 4.1, so formula (5.1) provides an extension of the usual spherical transform.

Moreover it is easy to verify that  $\mathcal{G}(L^j\Lambda) = \xi^j \mathcal{G}\Lambda$  for every  $\Lambda$  in  $\mathcal{S}'_{\text{rad}}(H_n)$ .

With this notation Theorem 4.1 extends to radial tempered distributions in the following form.

**Corollary 5.1.** *The spherical transform  $\mathcal{G}$  is a topological isomorphism between the spaces  $\mathcal{S}'_{\text{rad}}(H_n)$  and  $\mathcal{S}'_0(\Sigma)$ .*

We now study the behavior of the spherical transform of radial compactly supported distributions.

**Proposition 5.2.** *Let  $\Lambda$  be a radial compactly supported distribution on  $H_n$ . Then*

$$(5.2) \quad \widehat{\Lambda} : (\xi, \lambda) \mapsto \langle \Lambda, \check{\Phi}_{\xi, \lambda} \rangle_{H_n}$$

*is a holomorphic function on  $\mathbb{C}^2$ . Moreover  $\widehat{\Lambda}\mu$  is in  $\mathcal{S}'_0(\Sigma)$  and*

$$\mathcal{G}\Lambda = \widehat{\Lambda}\mu ,$$

*i.e.,  $\mathcal{G}\Lambda$  coincides with the function  $\widehat{\Lambda}$ .*

*Proof.* Using Lemma 3.1 it is easy to prove that  $\widehat{\Lambda}$  is entire. For the second part, we first check that for every  $\psi$  in  $\mathcal{S}(\mathbb{R}^2)$ , the integral  $\int_{\Sigma} \widehat{\Lambda} \psi d\mu$  is absolutely convergent, and therefore  $\widehat{\Lambda}\mu$  is in  $\mathcal{S}'_0(\Sigma)$ . Indeed, if  $(\xi, \lambda) = (|\lambda|(2j+n), \lambda)$  is in  $\Sigma^*$ , for some  $m$  in  $\mathbb{N}$

$$|\widehat{\Lambda}(\xi, \lambda)| = |\langle \Lambda, \check{\Phi}_{\xi, \lambda} \rangle_{H_n}| \leq C \|\check{\Phi}_{\xi, \lambda}\|_{C^m(K)} ,$$

where  $K = \text{supp } \Lambda \subset \Sigma$ . By Lemma 3.2, the function  $\widehat{\Lambda}$  is slowly growing on  $\Sigma$  and so for every  $\psi$  in  $\mathcal{S}(\mathbb{R}^2)$ , the integral  $\int_{\Sigma} \widehat{\Lambda} \psi d\mu$  is absolutely convergent.

When  $g$  is in  $\mathcal{S}(\mathbb{R}^2)$ ,

$$\begin{aligned} \langle \widehat{\Lambda}\mu, \psi \rangle_{\mathbb{R}^2} &= \int_{\Sigma} \widehat{\Lambda}(\xi, \lambda) \psi(\xi, \lambda) d\mu \\ &= \frac{1}{(2\pi)^{n+1}} \int_{\mathbb{R}} \sum_{j=0}^{\infty} \binom{j+n-1}{j} \widehat{\Lambda}(|\lambda|(2j+n), \lambda) \psi(|\lambda|(2j+n), \lambda) |\lambda|^n d\lambda \\ &= \lim_{N \rightarrow \infty} \frac{1}{(2\pi)^{n+1}} \int_{-N}^N \sum_{j=0}^N \binom{j+n-1}{j} \widehat{\Lambda}(|\lambda|(2j+n), \lambda) \psi(|\lambda|(2j+n), \lambda) |\lambda|^n d\lambda \\ &= \lim_{N \rightarrow \infty} \frac{1}{(2\pi)^{n+1}} \int_{-N}^N \sum_{j=0}^N \binom{j+n-1}{j} \langle \Lambda, \check{\Phi}_{|\lambda|(2j+n), \lambda} \rangle_{H_n} \psi(|\lambda|(2j+n), \lambda) |\lambda|^n d\lambda \\ &= \lim_{N \rightarrow \infty} \frac{1}{(2\pi)^{n+1}} \left\langle \Lambda, \int_{-N}^N \sum_{j=0}^N \binom{j+n-1}{j} \check{\Phi}_{|\lambda|(2j+n), \lambda} \psi(|\lambda|(2j+n), \lambda) |\lambda|^n d\lambda \right\rangle_{H_n} . \end{aligned}$$



Since

$$\lim_{N \rightarrow \infty} \frac{1}{(2\pi)^{n+1}} \int_{-N}^N \sum_{j=0}^N \check{\Phi}_{|\lambda|(2j+n), \lambda} \psi(|\lambda|(2j+n), \lambda) |\lambda|^n d\lambda = (\mathcal{G}^{-1}\psi|_{\Sigma})^\vee$$

uniformly on compacta and the same holds for all derivatives,

$$\langle \widehat{\Lambda} \mu, \psi \rangle_{\mathbb{R}^2} = \langle \Lambda, (\mathcal{G}^{-1}\psi|_{\Sigma})^\vee \rangle_{H_n} = \langle \mathcal{G}\Lambda, \psi \rangle_{\mathbb{R}^2} .$$

□

## 6. THE OPERATORS $M_{\pm}$

Denote by  $M_{\pm}$  the operators acting on a smooth function  $\psi$  on  $\mathbb{R}^2$  by the rule [8]

$$\begin{aligned} M_{\pm}\psi(\xi, \lambda) &= \partial_{\lambda}\psi(\xi, \lambda) \mp n\partial_{\xi}\psi(\xi, \lambda) - 2(n\lambda \pm \xi) \int_0^1 \partial_{\xi}^2\psi(\xi \pm 2\lambda t, \lambda)(1-t) dt. \\ &= \frac{1}{\lambda} (\lambda\partial_{\lambda} + \xi\partial_{\xi}) \psi(\xi, \lambda) - \frac{n\lambda \pm \xi}{2\lambda^2} (\psi(\xi \pm 2\lambda, \lambda) - \psi(\xi, \lambda)). \end{aligned}$$

Since  $\lambda\partial_{\lambda} + \xi\partial_{\xi}$  is the derivative in the radial direction, the operators  $M_{\pm}$  depend only on the restriction to the Heisenberg fan.

The operators  $M_{\pm}$  have the following relevant property. If  $f$  is radial and  $(1 + \mathcal{A})f$  is integrable on  $H_n$  then (see [8])

$$(6.1) \quad \mathcal{G}(\mathcal{A}f) = M_+(\mathcal{G}f) \quad \text{and} \quad \mathcal{G}(\bar{\mathcal{A}}f) = -M_-(\mathcal{G}f).$$

One can verify that

$$\langle M_+(\mathcal{G}f)\mu, \mathcal{G}h \rangle_{\mathbb{R}^2} = -\langle (\mathcal{G}f)\mu, M_-(\mathcal{G}h) \rangle_{\mathbb{R}^2} \quad \forall f, h \in \mathcal{S}_{\text{rad}}(H_n).$$

Hence by Theorem 4.1

$$(6.2) \quad \int_{\Sigma} (M_+\varphi) \psi d\mu = - \int_{\Sigma} \varphi (M_-\psi) d\mu \quad \forall \varphi, \psi \in \mathcal{S}(\mathbb{R}^2).$$

According to (6.2), when  $U$  is in  $\mathcal{S}'_0(\Sigma)$  we define the distribution  $M_+U$  by

$$\langle M_+U, \psi \rangle_{\mathbb{R}^2} = -\langle U, M_- \psi \rangle_{\mathbb{R}^2} \quad \forall \psi \in \mathcal{S}(\mathbb{R}^2)$$

and similarly we define  $M_-U$  with  $M_+$  and  $M_-$  interchanged.

Clearly if  $\psi|_{\Sigma} = 0$  then  $(M_- \psi)|_{\Sigma} = 0$ , so that  $M_+U$  is in  $\mathcal{S}'_0(\Sigma)$ .

Moreover it is easy to verify that (6.1) extends to distributions, i.e.

$$(6.3) \quad \mathcal{G}(\mathcal{A}\Lambda) = M_+(\mathcal{G}\Lambda) \quad \text{and} \quad \mathcal{G}(\bar{\mathcal{A}}\Lambda) = -M_-(\mathcal{G}\Lambda) \quad \forall \Lambda \in \mathcal{S}'(H_n).$$

Finally, given a distribution in  $\mathcal{S}'_0(\Sigma)$  of the form  $F\mu$ , where  $F$  is smooth and slowly growing on  $\mathbb{R}^2$ , we note that for all  $\psi$  in  $\mathcal{S}(\mathbb{R}^2)$ ,

$$\begin{aligned} \langle M_+(F\mu), \psi \rangle_{\mathbb{R}^2} &= -\langle F\mu, M_- \psi \rangle_{\mathbb{R}^2} \\ &= -\int_{\Sigma} F M_- \psi d\mu \\ &= \int_{\Sigma} M_+ F \psi d\mu = \langle (M_+ F) \mu, \psi \rangle_{\mathbb{R}^2}, \end{aligned}$$

therefore

$$(6.4) \quad M_+(F\mu) = (M_+ F) \mu .$$

For later use, we prove the following estimate.

**Lemma 6.1.** *Let  $a$  be a positive integer, then for every  $\psi$  in  $\mathcal{D}(\mathbb{R}^2)$  with support in the set  $\{(\xi, \lambda) \in \mathbb{R}^2 : |\xi| \leq \rho\}$*

$$\|M_{\pm}^a \psi\|_{L^1(\Sigma)} \leq C_a (1 + \rho)^{a+n+2} \sum_{s,r=0}^{2a} \|\partial_{\lambda}^s \partial_{\xi}^r \psi\|_{L^{\infty}(\mathbb{R}^2)}.$$

*Proof.* It is enough to prove the statement for  $M_+$ , since  $M_- \check{\psi} = -[M_+ \psi]^\sim$ , where  $\check{\psi}(\xi, \lambda) = \psi(\xi, -\lambda)$ . Let  $W$  denote the operator acting on a smooth function  $\psi$  on  $\mathbb{R}^2$  by

$$W\psi(\xi, \lambda) = 2 \int_0^1 \partial_\xi^2 \psi(\xi + 2\lambda t, \lambda) (1-t) dt.$$

For every  $j \geq 0$  let  $\eta_j$  be the function and let  $V_j$  be the operator defined by

$$\eta_j(\xi, \lambda) = \xi + (2j + n)\lambda \quad V_j = \partial_\lambda - (2j + n)\partial_\xi.$$

With this notation  $M_+ = V_0 - \eta_0 W$ . Moreover, as proved in [6, Lemma 4.5], for every positive integer  $a$ ,

$$(6.5) \quad M_+^a = V_0^a + \sum_{k=1}^a \eta_0 \cdots \eta_{k-1} D_{k,a},$$

where  $D_{k,a}$  is a polynomial in  $V_0, \dots, V_k, W$  of degree  $a$  such that in each monomial the operator  $W$  appears  $k$  times.

Let  $\psi$  be in  $\mathcal{D}(\mathbb{R}^2)$  with support in the set  $\{(\xi, \lambda) \in \mathbb{R}^2 : |\xi| \leq \rho\}$ . Then it is easy to see that  $\text{supp } D_{k,a}\psi \subseteq \{(\xi, \lambda) \in \mathbb{R}^2 : |\xi| \leq c\rho\}$ , with  $c$  depending on  $a$ . Therefore, using (4.1),

$$\begin{aligned} \|M_+^a \psi\|_{L^1(\Sigma)} &\leq \|V_0^a \psi\|_{L^1(\Sigma)} + \sum_{k=1}^a \|\eta_0 \cdots \eta_{k-1} D_{k,a} \psi\|_{L^1(\Sigma)} \\ &\leq C_a (1 + \rho)^{a+n+2} \left( \sum_{r+s \leq a} \|\partial_\lambda^s \partial_\xi^r \psi\|_{L^\infty(\mathbb{R}^2)} + \sum_{k=1}^a \|D_{k,a} \psi\|_{L^\infty(\mathbb{R}^2)} \right). \end{aligned}$$

We complete the proof by showing that

$$\|D_{k,a} \psi\|_{L^\infty(\mathbb{R}^2)} \leq C_a \sum_{s+r \leq 2a} \|\partial_\lambda^s \partial_\xi^r \psi\|_{L^\infty(\mathbb{R}^2)} \quad k = 1, 2, \dots, a,$$

by induction on  $a$ . Indeed, the case  $a = 1$  is trivial since

$$\|W\psi\|_{L^\infty(\mathbb{R}^2)} \leq 2 \int_0^1 \|\partial_\xi^2 \psi\|_{L^\infty(\mathbb{R}^2)} (1-t) dt \leq C \|\partial_\xi^2 \psi\|_{L^\infty(\mathbb{R}^2)}.$$

If  $a > 1$  then either  $D_{k,a} = D_{k-1,a-1} W$  or  $D_{k,a} = D_{k,a-1} V_j$ , for some  $j$  and  $k \leq a-1$ . The second case is trivial. If  $D_{k,a} = D_{k-1,a-1} W$ , we note that by induction on  $s$  it is easy to verify that

$$\partial_\lambda^s \partial_\xi^r W \psi = 2 \sum_{k=0}^s \binom{s}{k} \int_0^1 \partial_\lambda^{s-k} \partial_\xi^{r+2+k} \psi(\xi + 2\lambda t, \lambda) (2t)^k (1-t) dt.$$

Therefore

$$\begin{aligned} \|D_{k-1,a-1} W \psi\|_{L^\infty(\mathbb{R}^2)} &\leq C_a \sum_{s+r \leq 2a-2} \|\partial_\lambda^s \partial_\xi^r W \psi\|_{L^\infty(\mathbb{R}^2)} \\ &\leq C_a \sum_{s+r \leq 2a-2} \sum_{k=0}^s \|\partial_\lambda^{s-k} \partial_\xi^{r+2+k} \psi\|_{L^\infty(\mathbb{R}^2)} \\ &\leq C_a \sum_{s+r \leq 2a} \|\partial_\lambda^s \partial_\xi^r \psi\|_{L^\infty(\mathbb{R}^2)}. \end{aligned} \quad \square$$

## 7. REAL PALEY–WIENER RESULTS FOR THE SPHERICAL TRANSFORM

Suppose that  $\Lambda$  is a radial tempered distribution on  $H_n$ . Motivated by [1], we define  $R(\Lambda)$  in  $[0, \infty]$  by

$$R(\Lambda) = \max \{|x| : x \in \text{supp } \Lambda\}$$

and we call  $R(\Lambda)$  the radius of the support of the distribution  $\Lambda$ .

The purpose of this section is to prove real Paley–Wiener Theorems for the spherical transform; we start with a characterization of compactly supported radial distributions and then we specialize these results to square integrable radial functions and Schwartz radial functions. When a distribution  $U$  on  $\mathbb{R}^2$  is of the form  $U = F_U \mu$  with  $F_U$  a (smooth) function on  $\mathbb{R}^2$ , by abuse of notation we shall also denote by  $U$  the associated function  $F_U$ .

Our first characterization reads as follows.

**Theorem 7.1.** *Let  $\Lambda$  be in  $\mathcal{S}'_{\text{rad}}(H_n)$ . The following conditions are equivalent.*

- (1)  $R(\Lambda)$  is finite;
- (2)  $\mathcal{G}\Lambda$  is the restriction to  $\Sigma$  of a smooth function on  $\mathbb{R}^2$  and for every  $p$  in  $[1, \infty]$  there exists  $\beta > 0$  such that

$$\limsup_{j \rightarrow \infty} \|(1 + \xi)^{-\beta} M_+^j \mathcal{G}\Lambda\|_{L^p(\Sigma)}^{1/j} < \infty;$$

- (3) for every large  $j$  the distribution  $M_+^j \mathcal{G}\Lambda$  is the restriction to  $\Sigma$  of a smooth function on  $\mathbb{R}^2$  and there exist  $\beta > 0$  and  $p$  in  $[1, \infty]$  such that

$$\liminf_{j \rightarrow \infty} \|(1 + \xi)^{-\beta} M_+^j \mathcal{G}\Lambda\|_{L^p(\Sigma)}^{1/j} < \infty.$$

Moreover, if any of these conditions is satisfied, then for every  $p$  in  $[1, \infty]$  there exists  $\beta > 0$  such that

$$(7.1) \quad \lim_{j \rightarrow \infty} \|(1 + \xi)^{-\beta} M_+^j \mathcal{G}\Lambda\|_{L^p(\Sigma)}^{1/j} = R(\Lambda)^2.$$

Since  $M_- \psi = -[M_+ \check{\psi}]^\sim$ , where  $\check{\psi}(\xi, \lambda) = \psi(\xi, -\lambda)$ , we have also a corresponding analogue with  $M_-$  in place of  $M_+$ .

The proof of Theorem 7.1 is given after some preliminary results, the first of which is the following technical lemma.

**Lemma 7.2.** *Let  $R > 0$  and  $j$  be a positive integer. Suppose that  $f$  is a smooth function on  $H_n$  with compact support in the set  $\{x \in H_n : |x| > R\}$  and let  $f_j = \bar{\mathcal{A}}^{-j} f$ .*

*Then for every  $N$  in  $\mathbb{N}$*

$$\|(1 + \xi)^N \mathcal{G}f_j\|_{L^\infty(\Sigma)} \leq C_N j^{2N} R^{-2j} \max_{h \leq N} \sum_{k + \deg J = 2h} \|\bar{\mathcal{A}}^{-k} D^J f\|_{L^1(H_n)}$$

*Proof.* Note that since  $f$  is supported away from the origin, the function  $f_j = \bar{\mathcal{A}}^{-j} f$  is again smooth and compactly supported. Moreover,

$$\|(1 + \xi)^N \mathcal{G}f_j\|_{L^\infty(\Sigma)} = \|\mathcal{G}((I + L)^N f_j)\|_{L^\infty(\Sigma)}$$

$$\leq \|(I + L)^N f_j\|_{L^1(H_n)}.$$

Clearly  $(I + L)^N f_j = \sum \binom{M}{h} L^h f_j$  and by the Leibniz rule

$$\begin{aligned} \|(I + L)^N f_j\|_{L^1(H_n)} &\leq C_N \max_{h \leq N} \|L^h f_j\|_{L^1(H_n)} \\ &\leq C_N \max_{h \leq N} \sum_{\deg I + \deg J = 2h} \|(D^I \bar{\mathcal{A}}^{-j})(D^J f)\|_{L^1(H_n)} \\ &\leq C_N \max_{h \leq N} \sum_{\deg I + \deg J = 2h} j^{|I|} \|(\bar{\mathcal{A}}^{-j - \deg I})(D^J f)\|_{L^1(H_n)} \\ &\leq C_N j^{2N} R^{-2j} \max_{h \leq N} \sum_{\deg I + \deg J = 2h} \|\bar{\mathcal{A}}^{-\deg I} D^J f\|_{L^1(H_n)}. \quad \square \end{aligned}$$

Now we note that the spherical transform of radial compactly supported distributions satisfies a pointwise estimate on the Heisenberg fan  $\Sigma$ .

**Proposition 7.3.** *Let  $\Lambda$  be a radial compactly supported distribution of order  $N$  on  $H_n$ . Then for every  $R > R(\Lambda)$  there exists a constant  $C = C_R > 0$  such that for every  $j$  in  $\mathbb{N}$*

$$(7.2) \quad |M_+^j \widehat{\Lambda}(\xi, \lambda)| \leq C R^{2j} (1 + \xi)^{N/2} \quad \forall (\xi, \lambda) \in \Sigma.$$

*Proof.* We have already proved in Proposition 5.2 that  $\mathcal{G}\Lambda = \widehat{\Lambda}\mu$  and that  $\widehat{\Lambda}$  extends to an entire function, so  $\widehat{\Lambda}$  is in  $C^\infty(\Sigma)$ . Moreover by equations (6.3) and (6.4)

$$(M_+^j \widehat{\Lambda})\mu = M_+^j(\widehat{\Lambda}\mu) = M_+^j \mathcal{G}\Lambda = \mathcal{G}(\mathcal{A}^j \Lambda) = \widehat{\mathcal{A}^j \Lambda} \mu,$$

therefore  $M_+^j \widehat{\Lambda} = \widehat{\mathcal{A}^j \Lambda}$ .

Let  $R > R(\Lambda)$  and choose  $R_1$  such that  $R > R_1 > R(\Lambda)$ . Suppose that  $g$  is a radial test function on  $H_n$  such that  $g(x) = 1$  when  $x$  is in the support of  $\Lambda$  and  $g(x) = 0$  if  $|x| > R_1$ .

Then for all  $(\xi, \lambda)$  in  $\Sigma^*$ ,

$$\begin{aligned}
|M_+^j \widehat{\Lambda}(\xi, \lambda)| &= |\widehat{\mathcal{A}^j \Lambda}(\xi, \lambda)| = |\widehat{g \mathcal{A}^j \Lambda}(\xi, \lambda)| \\
&= |\langle g \mathcal{A}^j \Lambda, \check{\Phi}_{\xi, \lambda} \rangle_{H_n}| \\
&= |\langle \Lambda, g \mathcal{A}^j \Phi_{\xi, -\lambda} \rangle_{H_n}| \\
&\leq C \sum_{\deg I \leq N} \|D^I(g \mathcal{A}^j \Phi_{\xi, -\lambda})\|_{L^\infty(H_n)}.
\end{aligned}$$

We conclude by the Leibniz rule and Lemma 3.2 that

$$|M_+^j \widehat{\Lambda}(\xi, \lambda)| \leq C (1+j)^N R_1^{2j} (1+\xi)^{N/2} \quad \forall (\xi, \lambda) \in \Sigma^*,$$

which, since  $R > R_1$  and by the smoothness of  $\widehat{\Lambda}$ , implies (7.2).  $\square$

Conversely, it is easy to deduce that a radial tempered distribution is compactly supported when a certain limit is finite.

**Proposition 7.4.** *Let  $\Lambda$  be in  $\mathcal{S}'_{\text{rad}}(H_n)$ . Suppose that there exists  $J$  in  $\mathbb{N}$  such that for every  $j \geq J$  the distribution  $M_+^j \mathcal{G}\Lambda$  is of the form  $G_j \mu$ , where  $G_j$  is a locally integrable function with respect to  $\mu$ . Then for every  $N$  in  $\mathbb{N}$  and every  $p$  in  $[1, \infty]$*

$$\liminf_{j \rightarrow \infty} \| (1+\xi)^{-N} G_j \|_{L^p(\Sigma)}^{1/j} \geq R(\Lambda)^2.$$

*Proof.* Suppose that  $R(\Lambda) > 0$  and let  $0 < \varepsilon < R(\Lambda)/2$ . Then we may find a smooth function  $f$  with compact support in the set

$$\{x \in H_n : R(\Lambda) - \varepsilon < |x| < R(\Lambda) + \varepsilon\}$$

such that  $\langle \Lambda, \check{f} \rangle_{H_n} \neq 0$ . As in the previous lemma, the function  $f$  is supported away from the origin and we let  $f_j = \bar{\mathcal{A}}^{-j} f$ . By (5.1) and (6.3)

$$\begin{aligned} |\langle \Lambda, \check{f} \rangle_{H_n}| &= |\langle \Lambda, \mathcal{A}^j \mathcal{A}^{-j} \check{f} \rangle_{H_n}| = |\langle \Lambda, \mathcal{A}^j \check{f}_j \rangle_{H_n}| \\ &= |\langle \mathcal{A}^j \Lambda, \check{f}_j \rangle_{H_n}| = |\langle \mathcal{G}(\mathcal{A}^j \Lambda), \mathcal{G}f_j \rangle_{\mathbb{R}^2}| \\ &= |\langle M_+^j \mathcal{G}\Lambda, \mathcal{G}f_j \rangle_{\mathbb{R}^2}| = |\langle G_j \mu, \mathcal{G}f_j \rangle_{\mathbb{R}^2}| \\ &\leq \|(1 + \xi)^{-N} G_j\|_{L^p(\Sigma)} \|(1 + \xi)^N \mathcal{G}f_j\|_{L^{p'}(\Sigma)}. \end{aligned}$$

In the case where  $\|(1 + \xi)^{-N} G_j\|_{L^p(\Sigma)} = \infty$  for all  $j$ , there is nothing to prove. Otherwise, since  $|\langle \Lambda, \check{f} \rangle_{H_n}| \neq 0$ ,

$$\liminf_{j \rightarrow \infty} \|(1 + \xi)^{-N} G_j\|_{L^p(\Sigma)}^{1/j} \geq \liminf_{j \rightarrow \infty} \left( \frac{|\langle \Lambda, \check{f} \rangle_{H_n}|}{\|(1 + \xi)^N \mathcal{G}f_j\|_{L^{p'}(\Sigma)}} \right)^{1/j} = \liminf_{j \rightarrow \infty} \|(1 + \xi)^N \mathcal{G}f_j\|_{L^{p'}(\Sigma)}^{-1/j}.$$

Since there exists  $M$  in  $\mathbb{N}$  such that  $\xi \mapsto (1 + \xi)^{N-M}$  is in  $L^{p'}(\Sigma)$ , by Lemma 7.2 we conclude that

$$\begin{aligned} \|(1 + \xi)^N \mathcal{G}f_j\|_{L^{p'}(\Sigma)} &\leq \|(1 + \xi)^{N-M}\|_{L^{p'}(\Sigma)} \|(1 + \xi)^M \mathcal{G}f_j\|_{L^\infty(\Sigma)} \\ &\leq C j^{2M} (R(\Lambda) - \varepsilon)^{-2j}, \end{aligned}$$

and the thesis follows easily.

When  $R(\Lambda) = \infty$  we use the same arguments to show that  $\liminf_{j \rightarrow \infty} \|(1 + \xi)^{-N} G_j\|_{L^p(\Sigma)}^{1/j} \geq R$  for every  $R > 0$ .  $\square$

Putting together Proposition 7.3 and Proposition 7.4, we obtain the following criterion, by which we can measure the size of the support of a radial compactly supported distribution.

**Corollary 7.5.** *Let  $\Lambda$  be a radial compactly supported distribution of order  $N$ . Then*

$$\lim_{j \rightarrow \infty} \|(1 + \xi)^{-N/2} M_+^j \widehat{\Lambda}\|_{L^\infty(\Sigma)}^{1/j} = R(\Lambda)^2.$$



*Proof.* From the pointwise estimate (7.2), we deduce that for every  $R > R(\Lambda)$

$$\limsup_{j \rightarrow \infty} \|(1 + \xi)^{-N/2} M_+^j \widehat{\Lambda}\|_{L^\infty(\Sigma)}^{1/j} \leq R^2,$$

therefore  $\limsup_{j \rightarrow \infty} \|(1 + \xi)^{-N/2} M_+^j \widehat{\Lambda}\|_{L^\infty(\Sigma)}^{1/j} \leq R(\Lambda)^2$ . The thesis follows by Proposition 7.4.  $\square$

*Proof of Theorem 7.1.* If  $\Lambda$  is compactly supported and of order  $N$  then by Proposition 5.2 it coincides with a smooth slowly growing function  $G$  on  $\mathbb{R}^2$ . If  $\beta > 0$  is such that  $(1 + \xi)^{N/2-\beta}$  is in  $L^p(\Sigma)$ , we have

$$(7.3) \quad \|(1 + \xi)^{-\beta} M_+^j G\|_{L^p(\Sigma)} \leq \|(1 + \xi)^{N/2-\beta}\|_{L^p(\Sigma)} \|(1 + \xi)^{-N/2} M_+^j G\|_{L^\infty(\Sigma)}.$$

Hence by Corollary 7.5 we have that (1)  $\Rightarrow$  (2). The implication (2)  $\Rightarrow$  (3) is trivial and the implication (3)  $\Rightarrow$  (1) is a consequence of Proposition 7.4. Finally (7.1) follows by (7.3), Proposition 7.4 and Corollary 7.5.  $\square$

### 7.1. Square-integrable functions.

**Theorem 7.6.** *Suppose that for every  $j \geq 0$  the function  $M_+^j \psi$  is in  $L^2(\Sigma)$ . Then the function  $f$  such that  $\mathcal{G}f = \psi$  is in  $L^2(H_n)$  and*

$$\lim_{j \rightarrow \infty} \|M_+^j \psi\|_{L^2(\Sigma)}^{1/j} = R(f)^2.$$

*Proof.* By Proposition 7.4 it is enough to check that  $\limsup_{j \rightarrow \infty} \|M_+^j \psi\|_{L^2(\Sigma)}^{1/j} \leq R(f)^2$ , and this is easily established by using the Plancherel formula. Indeed, when  $R(f)$  is finite,

$$\|M_+^j \psi\|_{L^2(\Sigma)} = \|\mathcal{A}^j f\|_{L^2(H_n)} \leq R(f)^{2j} \|f\|_{L^2(H_n)}. \quad \square$$

**Corollary 7.7.** *Let  $R \geq 0$ . Then  $\mathcal{G}$  is a bijection from the space  $L_{\text{rad}, R}^2(H_n)$  of square integrable radial functions  $f$  such that  $R(f) \leq R$  onto  $\{\psi \in L^2(\Sigma) : \lim_{j \rightarrow \infty} \|M_+^j \psi\|_{L^2(\Sigma)}^{1/j} \leq R^2\}$ .*

**7.2. Schwartz functions.** The purpose of this subsection is to prove the following characterization.

**Theorem 7.8.** *Let  $f$  be in  $\mathcal{S}_{\text{rad}}(H_n)$ . The following conditions are equivalent.*

- (1)  $R(f)$  is finite;
- (2) for every  $h \geq 0$  and every  $p$  in  $[1, \infty]$ ,  $\limsup_{j \rightarrow \infty} \|\xi^h M_+^j \mathcal{G}f\|_{L^p(\Sigma)}^{1/j}$  is finite;
- (3) there exists  $p$  in  $[1, \infty]$  such that  $\liminf_{j \rightarrow \infty} \|M_+^j \mathcal{G}f\|_{L^p(\Sigma)}^{1/j}$  is finite.

Moreover, if any of these conditions is satisfied, then for every  $h \geq 0$  and every  $p$  in  $[1, \infty]$ ,

$$\lim_{j \rightarrow \infty} \|(1 + \xi)^h M_+^j \mathcal{G}f\|_{L^p(\Sigma)}^{1/j} = R(f)^2.$$

Note that the implication (2)  $\Rightarrow$  (3) is trivial, and that (3)  $\Rightarrow$  (1) follows from Proposition 7.4. In the next proposition we prove the implication (1)  $\Rightarrow$  (2).

**Proposition 7.9.** *Suppose that  $f$  is a radial Schwartz function on  $H_n$ . Then for every  $h \geq 0$  and every  $p$  in  $[1, \infty]$*

$$\limsup_{j \rightarrow \infty} \|(1 + \xi)^h M_+^j \mathcal{G}f\|_{L^p(\Sigma)}^{1/j} \leq R(f)^2.$$

*Proof.* If  $R(f) = \infty$  there is nothing to prove. If  $R(f) = 0$ , then  $f = 0$  and the conclusion is again trivial. We therefore suppose that  $R(f)$  is positive. Note that

$$\xi^h M_+^j \mathcal{G}f(\lambda, \xi) = \mathcal{G}(L^h \mathcal{A}^j f)$$

and when  $j \geq 2h$ , by the Leibniz rule

$$\begin{aligned} |L^h \mathcal{A}^j f| &= \left| \sum_{\deg I + \deg J = 2h} c_{h,I,J} (D^I \mathcal{A}^j) (D^J f) \right| \\ &\leq \sum_{\deg I + \deg J = 2h} |c_{h,I,J}| j^{|I|} |\mathcal{A}^{j-\deg I}| |D^J f|. \end{aligned}$$

Therefore

$$\begin{aligned}
\| \xi^h M_+^j \mathcal{G} f \|_{L^\infty(\Sigma)} &\leq \| L^h \mathcal{A}^j f \|_{L^1(H_n)} \\
&\leq C_h j^{2h} \sum_{q \leq 2h} \max_{\deg J = 2h-q} \| \mathcal{A}^{j-q} D^J f \|_{L^1(H_n)} \\
&\leq C_h j^{2h} \sum_{q \leq 2h} R(f)^{2j-2q} \max_{\deg J = 2h-q} \| D^J f \|_{L^1(H_n)} \\
&= C_{f,h} j^{2h} R(f)^{2j}.
\end{aligned}$$

We note that for a sufficiently big integer  $M$  the function  $(\lambda, \xi) \mapsto (1 + \xi)^{-M}$  is in  $L^p(\Sigma)$ , so that

$$\begin{aligned}
\| (1 + \xi)^h M_+^j \mathcal{G} f \|_{L^p(\Sigma)} &\leq C \| (1 + \xi)^{M+h} M_+^j \mathcal{G} f \|_{L^\infty(\Sigma)} \\
&\leq C_{f,M,h} (1 + j^{2M+2h}) R(f)^{2j},
\end{aligned}$$

and taking the  $j$ -th root, the desired inequality follows.  $\square$

**Corollary 7.10.** *Suppose that  $f$  is a radial Schwartz function on  $H_n$  and let  $1 \leq p \leq \infty$ . Then for every  $h$  in  $\mathbb{N}$*

$$\lim_{j \rightarrow \infty} \| (1 + \xi)^h M_+^j \mathcal{G} f \|_{L^p(\Sigma)}^{1/j} = R(f)^2.$$

*Proof.* Since  $\| (1 + \xi)^h M_+^j \mathcal{G} f \|_{L^p(\Sigma)} \geq \| M_+^j \mathcal{G} f \|_{L^p(\Sigma)}$ , by Proposition 7.4 we obtain

$$\liminf_{j \rightarrow \infty} \| (1 + \xi)^h M_+^j \mathcal{G} f \|_{L^p(\Sigma)}^{1/j} \geq \liminf_{j \rightarrow \infty} \| M_+^j \mathcal{G} f \|_{L^p(\Sigma)}^{1/j} \geq R(f)^2.$$

The thesis follows from Proposition 7.9.  $\square$

## 8. PALEY–WIENER THEOREMS FOR THE INVERSE SPHERICAL TRANSFORM

In this section we describe the inverse spherical transform of compactly supported distributions in  $\mathcal{S}'_0(\Sigma)$ .

Given a compactly supported distribution in  $\mathcal{S}'(\mathbb{R}^2)$ , we define the function  $f_U$  on the Heisenberg group by

$$f_U(z, t) = \langle U, \Phi_{(\cdot)}(z, t) \rangle_{\mathbb{R}^2} \quad \forall (z, t) \in H_n.$$

An easy consequence of Lemma 3.1 is the following.

**Lemma 8.1.** *Let  $U$  be a compactly supported distribution in  $\mathcal{S}'(\mathbb{R}^2)$ . Then the function*

$$(x, y, t) \mapsto f_U(x + iy, t) = \langle U, \Phi_{(\cdot)}(x + iy, t) \rangle_{\mathbb{R}^2}$$

*extends to a holomorphic function on  $\mathbb{C}^{2n+1}$ .*

If  $U$  is in  $\mathcal{S}'(\mathbb{R}^2)$ , define

$$\rho(U) = \max \{ |\xi| : (\xi, \lambda) \in \text{supp } U \},$$

so that a distribution  $U$  in  $\mathcal{S}'_0(\Sigma)$  is compactly supported if and only if  $\rho(U)$  is finite.

In the next proposition we prove that if  $U$  is a compactly supported distribution in  $\mathcal{S}'_0(\Sigma)$  then the function  $f_U$  is a slowly growing function on  $H_n$  and it coincides with the inverse spherical transform of  $U$ .

**Proposition 8.2.** *Let  $U$  be a compactly supported distribution in  $\mathcal{S}'_0(\Sigma)$  and let, as before,*

$$f_U(z, t) = \langle U, \Phi_{(\cdot)}(z, t) \rangle_{\mathbb{R}^2} \quad \forall (z, t) \in H_n.$$

Then  $U = \mathcal{G}(f_U m)$  and  $f_U$  is a slowly growing function on  $H_n$  together with all its derivatives.

Moreover, for every  $\rho > \rho(U)$  there exist  $C = C_\rho$  and  $M$  such that for all  $j \geq 0$

$$(8.1) \quad |L^j f_U(z, t)| \leq C (1 + j)^k \rho^j (1 + |(z, t)|)^M \quad \forall (z, t) \in H_n,$$

where  $k$  is the order of  $U$ .

**Remark 8.3.** Observe that if  $U$  is a distribution in  $\mathcal{S}'(\mathbb{R}^2)$  with compact support in  $\Sigma$ , then the function  $f_U$  may not be slowly growing. Indeed let  $U = \partial_\xi \delta_{(n,1)}$  where  $\delta_{(n,1)}$  is the Dirac measure at the point  $(n, 1)$  in  $\Sigma$ . Then, reasoning as in Lemma 3.1, when  $(z, t)$  is in  $H_n$

$$f_U(z, t) = -\partial_\xi \Phi_{\xi, \lambda|_{(n,1)}}(z, t) = \frac{e^{it} e^{-|z|^2/4}}{2} \sum_{k=1}^{\infty} \frac{(|z|^2/2)^k}{k (n)_k}.$$

Since  $k < k + n$  and  $(n)_k \leq (n + k - 1)!$  we obtain when  $|z|$  is large

$$|f_U(z, t)| > \frac{e^{-|z|^2/4}}{2} \sum_{k=1}^{\infty} \frac{(|z|^2/2)^k}{(n + k)!} \sim \frac{e^{|z|^2/4}}{2(|z|^2/2)^n}.$$

This is due, much as in Subsection 4.2, to the fact that the holomorphic extension of spherical functions does not satisfy good estimates away from the Heisenberg fan. The main point in the proof of Proposition 8.2 is that, according to formula (8.2), if  $U$  is in  $\mathcal{S}'_0(\Sigma)$  one is allowed to choose a different extension.

*Proof.* By Theorem 5.1 there exists  $\Lambda$  in  $\mathcal{S}'_{\text{rad}}(H_n)$  such that  $\mathcal{G}\Lambda = U$ . Let  $g$  be in  $\mathcal{D}(H_n)$ , then

$$\begin{aligned}
\langle f_U, g \rangle_{H_n} &= \int_{H_n} f_U(z, t) g(z, t) dz dt \\
&= \int_{H_n} \langle U, \Phi_{(\cdot)}(z, t) \rangle_{\mathbb{R}^2} g(z, t) dz dt \\
&= \left\langle U, \int_{H_n} \Phi_{(\cdot)}(z, t) g(z, t) dz dt \right\rangle_{\mathbb{R}^2} \\
&= \langle U, \mathcal{G}\check{g} \rangle_{\mathbb{R}^2} \\
&= \langle \Lambda, g \rangle_{H_n} .
\end{aligned}$$

Hence the distribution  $\Lambda$  coincides with the function  $f_U$ , which is smooth.

We first prove the estimate (8.1). Fix  $(z, t)$  in  $H_n$ . Let  $k$  be the order of  $U$ , let  $\rho > \rho(U)$  and denote by  $B_\rho$  the ball of radius  $\rho$  in  $\mathbb{R}^2$ . Then for every  $j \geq 0$

$$\begin{aligned}
|L^j f_U(z, t)| &= |\langle U, L^j \Phi_{(\cdot)}(z, t) \rangle_{\mathbb{R}^2}| = |\langle U, \xi^j \Phi_{(\cdot)}(z, t) \rangle_{\mathbb{R}^2}| \\
(8.2) \quad &\leq C \inf \left\{ \|\xi^j \varphi^{z,t}\|_{C^k(B_\rho)} : \varphi^{z,t} \in C^k(\mathbb{R}^2), \quad \varphi^{z,t}|_{\Sigma \cap B_\rho} = \Phi_{(\cdot)}(z, t) \right\} \\
&\leq C_\rho (1+j)^k \rho^j \inf \left\{ \|\varphi^{z,t}\|_{C^k(B_\rho)} : \varphi^{z,t} \in C^k(\mathbb{R}^2), \quad \varphi^{z,t}|_{\Sigma \cap B_\rho} = \Phi_{(\cdot)}(z, t) \right\}
\end{aligned}$$

In order to obtain the desired estimate we shall choose a suitable extension  $\varphi^{z,t}$  of  $\Phi_{(\cdot)}(z, t)$ .

Let  $\psi$  be a smooth function on  $\mathbb{R}^2$  with compact support such that  $\psi|_{B_\rho} = 1$ . By Theorem 4.1 there exists  $u$  in  $\mathcal{S}_{\text{rad}}(H_n)$  such that  $\mathcal{G}u = \psi|_\Sigma$ . If  $\nu_{(z,t)}$  denotes the measure defined by

$$\int_{H_n} g(w, s) d\nu_{(z,t)}(w, s) = \int_{U(n)} g(kz, t) dk \quad \forall g \in C_c(H_n),$$

then for every  $(\xi, \lambda)$  in  $B_\rho \cap \Sigma$

$$\Phi_{\xi, \lambda}(z, t) = \mathcal{G}\check{\nu}_{(z,t)}(\xi, \lambda) = \mathcal{G}\check{\nu}_{(z,t)}(\xi, \lambda) \psi(\xi, \lambda) = \mathcal{G}(\check{\nu}_{(z,t)} * u)(\xi, \lambda).$$

Since  $\check{\nu}_{(z,t)} * u$  belongs to  $\mathcal{S}_{\text{rad}}(H_n)$ , then by Theorem 4.1 there exist  $\varphi^{z,t}$  in  $\mathcal{S}(\mathbb{R}^2)$  and  $M \geq 0$  such that

$$\varphi^{z,t}(\xi, \lambda) = \mathcal{G}(\check{\nu}_{(z,t)} * u)(\xi, \lambda) \quad \forall (\xi, \lambda) \in \Sigma$$

and

$$\|\varphi^{z,t}\|_{C^k(B_\rho)} \leq C \|\check{\nu}_{(z,t)} * u\|_{(M)}.$$

Moreover

$$\varphi^{z,t}(\xi, \lambda) = \Phi_{(\xi, \lambda)}(z, t) \quad \forall (\xi, \lambda) \in B_\rho \cap \Sigma.$$

If  $\tau_{(w,s)}u(w', s') = u((w, s)^{-1}(w', s'))$  denotes the left translation, then

$$\begin{aligned} \|\varphi^{z,t}\|_{C^k(B_\rho)} &\leq C \|\nu_{(z,t)} * u\|_{(M)} \\ &= C \left\| \int_{H_n} \tau_{(w,s)}u \, d\nu_{(z,t)}(w, s) \right\|_{(M)} \\ &\leq C \int_{H_n} \|\tau_{(w,s)}u\|_{(M)} \, d\nu_{(z,t)}(w, s) \\ &\leq C \int_{H_n} (1 + |w|^4 + s^2)^{M/4} \, d\nu_{(z,t)}(w, s) \\ &= C (1 + |(z, t)|)^M. \end{aligned}$$

Therefore there exists  $M$  such that for all  $j \geq 0$

$$|L^j f_U(z, t)| \leq C (1 + j)^k \rho^j (1 + |(z, t)|)^M \quad \forall (z, t) \in H_n.$$

The proof above can be adapted to prove that for every differential operator  $D^I$  of the form (2.1) there exists  $M > 0$  such that

$$|D^I f_U(z, t)| \leq C (1 + |(z, t)|)^M \quad \forall (z, t) \in H_n.$$

Indeed, note that

$$D^I f_U(z, t) = \langle U, D^I \Phi_{(\cdot)}(z, t) \rangle_{\mathbb{R}^2},$$

therefore

$$|D^I f_U(z, t)| \leq C \inf \left\{ \|\varphi^{z,t,I}\|_{C^k(B_\rho)} : \varphi^{z,t,I} \in C^k(\mathbb{R}^2) \quad \varphi^{z,t,I}|_{\Sigma \cap B_\rho} = D^I \Phi(\cdot)(z, t) \right\}.$$

Fix  $(z, t)$  in  $H_n$  and consider the distribution  $D_{(z,t)}^I \nu_{(z,t)}$  defined by the rule

$$\langle D_{(z,t)}^I \nu_{(z,t)}, \varphi \rangle_{H_n} = D^I \left( \int_K \varphi(kz, t) dk \right)$$

Then  $D_{(z,t)}^I \nu_{(z,t)}$  is a radial distribution supported in the orbit of  $(z, t)$ , hence it has compact support. So, for  $\psi$  and  $u$  as above,  $D_{(z,t)}^I \check{\nu}_{(z,t)} * u$  is in  $\mathcal{S}_{\text{rad}}(H_n)$  and by [4, Proposition 3.2] there exists  $\varphi^{z,t,I}$  in  $C^k(\mathbb{R}^2)$  and  $M$  such that

$$\varphi_{|\Sigma}^{z,t,I} = \mathcal{G}(D_{(z,t)}^I \check{\nu}_{(z,t)} * u) \quad \|\varphi^{z,t,I}\|_{C^k(B_\rho)} \leq C \|D_{(z,t)}^I \check{\nu}_{(z,t)} * u\|_{(M)}$$

Since  $\mathcal{G}u|_{\Sigma \cap B_\rho} = \psi|_{\Sigma \cap B_\rho} = 1$ ,

$$\varphi^{z,t,I}(\xi, \lambda) = \mathcal{G}(D_{(z,t)}^I \check{\nu}_{(z,t)} * u)(\xi, \lambda) = \mathcal{G}(D_{(z,t)}^I \check{\nu}_{(z,t)})(\xi, \lambda) \quad \forall (\xi, \lambda) \in \Sigma \cap B_\rho,$$

and by Proposition 5.2

$$\begin{aligned} \mathcal{G}(D_{(z,t)}^I \check{\nu}_{(z,t)})(\xi, \lambda) &= \langle D_{(z,t)}^I \check{\nu}_{(z,t)}, \check{\Phi}_{\xi,\lambda} \rangle_{\mathbb{R}^2} = D^I \left( (w, s) \mapsto \int_K \Phi_{\xi,\lambda}(kw, s) dk \right) (z, t) \\ &= D^I \Phi_{\xi,\lambda}(z, t). \end{aligned}$$

Finally, reasoning as before,

$$\|\varphi^{z,t,I}\|_{C^k(B_\rho)} \leq C \|D_{(z,t)}^I \check{\nu}_{(z,t)} * u\|_{(M)} \leq C (1 + |(z, t)|)^M \quad \forall (z, t) \in H_n. \quad \square$$

Our characterization of the inverse spherical transform of compactly supported distributions is the following.

**Theorem 8.4.** *Let  $U$  be in  $\mathcal{S}'_0(\Sigma)$ . The following conditions are equivalent.*

- (1)  $\rho(U)$  is finite;



- (2)  $\mathcal{G}^{-1}U$  coincides with a smooth slowly growing function on  $H_n$  and for every  $p$  in  $[1, \infty]$  there exists  $\beta > 0$  such that

$$\limsup_{j \rightarrow \infty} \| (1 + \mathcal{A})^{-\beta} L^j \mathcal{G}^{-1}U \|_{L^p(H_n)}^{1/j} < \infty;$$

- (3) for every large  $j$  the distribution  $L^j \mathcal{G}^{-1}U$  coincides with a measurable function on  $H_n$  and there exist  $\beta > 0$  and  $p$  in  $[1, \infty]$  such that

$$\liminf_{j \rightarrow \infty} \| (1 + \mathcal{A})^{-\beta} L^j \mathcal{G}^{-1}U \|_{L^p(H_n)}^{1/j} < \infty.$$

Moreover, if any of these conditions is satisfied, then  $\mathcal{G}^{-1}U$  is a smooth slowly growing function on  $H_n$  and for every  $p$  in  $[1, \infty]$  there exists  $\beta > 0$  such that

$$(8.3) \quad \lim_{j \rightarrow \infty} \| (1 + \mathcal{A})^{-\beta} L^j \mathcal{G}^{-1}U \|_{L^p(H_n)}^{1/j} = \rho(U).$$

As in the previous section, we split the proof of our characterization into several parts.

**Proposition 8.5.** *Let  $U$  be in  $\mathcal{S}'_0(\Sigma)$ . Suppose that there exists  $J$  in  $\mathbb{N}$  such that for every  $j \geq J$  the distribution  $L^j \mathcal{G}^{-1}U$  is of the form  $f_j m$ , where  $f_j$  is a locally integrable function on  $H_n$ . Then for every  $N$  in  $\mathbb{N}$  and every  $p$  in  $[1, \infty]$*

$$\liminf_{j \rightarrow \infty} \| (1 + \mathcal{A})^{-N} f_j \|_{L^p(H_n)}^{1/j} \geq \rho(U).$$

*Proof.* For the reader's convenience we write the proof although it follows the lines of that of Proposition 7.4. We may suppose that  $\rho(U)$  is positive, because in the case where  $\rho(U) = 0$ , there is nothing to prove.

Let  $\| (1 + \mathcal{A})^{-N} f_j \|_{L^p(H_n)} < \infty$ . Suppose that  $0 < \varepsilon < \rho(U)/2$  and let  $\psi$  be smooth function on  $\mathbb{R}^2$  with compact support in the set

$$\{(\xi, \lambda) \in \mathbb{R}^2 : \rho(U) - \varepsilon < \xi < \rho(U) + \varepsilon\}$$

such that  $\langle U, \psi \rangle_{\mathbb{R}^2} \neq 0$ . For every nonnegative integer  $j$ , define a smooth function on  $\mathbb{R}^2$  with compact support by  $\psi_j(\xi, \lambda) = \xi^{-j} \psi(\xi, \lambda)$  for every  $(\xi, \lambda)$  in  $\mathbb{R}^2$ . Then for  $1 \leq p \leq \infty$ ,

$$\begin{aligned} |\langle U, \psi \rangle_{\mathbb{R}^2}| &= |\langle \xi^j U, \psi_j \rangle_{\mathbb{R}^2}| = |\langle L^j \mathcal{G}^{-1} U, (\mathcal{G}^{-1} \psi_j|_{\Sigma})^\vee \rangle_{H_n}| \\ &\leq \|(1 + \mathcal{A})^{-N} f_j\|_{L^p(H_n)} \|(1 + \mathcal{A})^N \mathcal{G}^{-1} \psi_j|_{\Sigma}\|_{L^{p'}(H_n)} \end{aligned}$$

Let  $a$  be a positive integer such that  $\|(1 + \mathcal{A})^{N-a}\|_{L^{p'}(H_n)} < \infty$ , then by Lemma 6.1

$$\begin{aligned} \|(1 + \mathcal{A})^N \mathcal{G}^{-1} \psi_j|_{\Sigma}\|_{L^{p'}(H_n)} &\leq \|(1 + \mathcal{A})^{N-a}\|_{L^{p'}(H_n)} \|(1 + M_+)^a \psi_j\|_{L^1(\Sigma)} \\ &\leq C_a (\rho(U) + \varepsilon)^a \sum_{s,r=1}^{2a} \|\partial_\lambda^s \partial_\xi^r \psi_j\|_{L^\infty(\mathbb{R}^2)} \\ &\leq C_a (\rho(U) + \varepsilon)^a j^{2a} (\rho(U) - \varepsilon)^{-j}. \end{aligned}$$

Therefore

$$\|(1 + \mathcal{A})^{-N} f_j\|_{L^p(H_n)} \geq |\langle U, \psi \rangle_{\mathbb{R}^2}| C_{a,\varepsilon} j^{-2a} (\rho(U) - \varepsilon)^j$$

and the thesis follows. Similar considerations can be used in the case where  $\rho(U) = \infty$ .  $\square$

**Proposition 8.6.** *Let  $U$  be in  $\mathcal{S}'_0(\Sigma)$  with  $\rho(U) < \infty$ . Then  $\mathcal{G}^{-1}U$  coincides with a smooth slowly growing function  $f$  on  $H_n$  and for every  $p$  in  $[1, \infty]$  there exists  $h > 0$  such that*

$$\limsup_{j \rightarrow \infty} \|(1 + \mathcal{A})^{-h} L^j f\|_{L^p(H_n)}^{1/j} \leq \rho(U).$$

*Proof.* Since  $\rho(U) < \infty$ , the distribution  $U$  is compactly supported and therefore  $\mathcal{G}^{-1}U$  coincides with the smooth function  $f_U$  on  $H_n$  by Lemma 8.1 and  $f_U$  is slowly growing by Proposition 8.2. Moreover, the estimate (8.1) holds: if  $\rho > \rho(U)$  and  $k$  is the degree of  $U$ , there exists  $M$  such that for all  $j \geq 0$

$$|L^j f(z, t)| \leq C (1 + j)^k \rho^j (1 + |(z, t)|)^M.$$

Let  $p$  in  $[1, \infty]$  be fixed and choose  $h$  such that  $(1 + \mathcal{A})^{-h+M/2}$  is in  $L^p(H_n)$ . Then for every  $\rho > \rho(U)$

$$\begin{aligned} \|(1 + \mathcal{A})^{-h} L^j f\|_{L^p(H_n)} &\leq \|(1 + \mathcal{A})^{-h+M/2}\|_{L^p(H_n)} \|(1 + \mathcal{A})^{-M/2} L^j f\|_{L^\infty(H_n)} \\ &\leq C(1 + j)^k \rho^j \end{aligned}$$

so that  $\limsup_{j \rightarrow \infty} \|(1 + \mathcal{A})^{-h} L^j f\|_{L^p(H_n)}^{1/j} \leq \rho$ , for every  $\rho > \rho(U)$ .  $\square$

**8.1. Square-integrable functions and Schwartz functions.** Reasoning as in the proof of Theorem 7.6 and Corollary 7.7 it easy to prove the following characterization for square-integrable functions.

**Theorem 8.7.** *Let  $\rho \geq 0$ . Then  $\mathcal{G}$  is a bijection from the space*

$$\{f \in L^2_{\text{rad}}(H_n) : \lim_{j \rightarrow \infty} \|L^j f\|_{L^2(H_n)}^{1/j} \leq \rho\}$$

*onto the space*

$$\{F \in L^2(\Sigma) : \rho(F) \leq \rho\}.$$

In the case of Schwartz functions, we obtain the following results. For  $F$  in  $\mathcal{S}(\Sigma)$  we denote  $\rho(F) = \rho(F\mu)$ , so that

$$\rho(F) = \sup\{\xi : F(\xi, \lambda) \neq 0 \quad \text{and} \quad (\xi, \lambda) \in \Sigma\}.$$

**Proposition 8.8.** *Let  $1 \leq p \leq \infty$  and let  $F$  be in  $\mathcal{S}(\Sigma)$ . Then for every  $h \geq 0$ ,*

$$\lim_{j \rightarrow \infty} \|(1 + \mathcal{A})^h L^j \mathcal{G}^{-1} F\|_{L^p(H_n)}^{1/j} = \rho(F).$$

*Proof.* Suppose  $0 < \rho(F) < \infty$ . If  $\gamma > 0$  is big enough so that  $(1 + \mathcal{A})^{-\gamma}$  is in  $L^p(H_n)$  by Lemma 6.1 we obtain

$$\|(1 + \mathcal{A})^h L^j \mathcal{G}^{-1} F\|_{L^p(H_n)} \leq \|(1 + \mathcal{A})^{-\gamma}\|_{L^p(H_n)} \|(1 + \mathcal{A})^{h+\gamma} L^j \mathcal{G}^{-1} F\|_{L^\infty(H_n)}$$

$$\begin{aligned}
&\leq C \| (1 + M_+)^{h+\gamma} (\xi^j F) \|_{L^1(\Sigma)} \\
&\leq C j^{2h+2\gamma} (\rho(F))^j.
\end{aligned}$$

Hence

$$\limsup_{j \rightarrow \infty} \| (1 + \mathcal{A})^h L^j \mathcal{G}^{-1} F \|_{L^p(H_n)}^{1/j} \leq \rho(F).$$

and the thesis follows from Proposition 8.5. The cases  $\rho(F) = 0, \infty$  are trivial.  $\square$

**Theorem 8.9.** *Let  $F$  be in  $\mathcal{S}(\Sigma)$ . The following conditions are equivalent.*

- (1)  $\rho(F)$  is finite;
- (2) for every  $h \geq 0$  and every  $p$  in  $[1, \infty]$ ,  $\limsup_{j \rightarrow \infty} \| \mathcal{A}^h L^j \mathcal{G}^{-1} F \|_{L^p(H_n)}^{1/j}$  is finite;
- (3) there exists  $p$  in  $[1, \infty]$  such that  $\liminf_{j \rightarrow \infty} \| L^j \mathcal{G}^{-1} F \|_{L^p(H_n)}^{1/j}$  is finite.

Moreover, if any of these conditions is satisfied, then for every  $h \geq 0$  and every  $p$  in  $[1, \infty]$ ,

$$\lim_{j \rightarrow \infty} \| (1 + \mathcal{A})^h L^j \mathcal{G}^{-1} F \|_{L^p(H_n)}^{1/j} = \rho(F).$$

*Proof.* The implication (1)  $\Rightarrow$  (2) follows by Proposition 8.8. The implication (2)  $\Rightarrow$  (3) is trivial. The implication (3)  $\Rightarrow$  (1) follows by Proposition 8.5.  $\square$

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